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A relativistic superalgebra in a generalized Schrödinger picture

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Abstract. We consider a relativistic superalgebra in the picture in which the time and spatial derivative cannot be presented in the operators of the particle. The supersymmetry generators as well as the Hamilton operators for the massive relativistic particles with spin 0 and spin 1/2 are expressed in terms of the principal series of the unitary representations of the Lorentz group. We also consider the massless case. New Hamilton operators are constructed for the massless particles with spin 0 and spin 1/2.

1 Introduction

The aim of the present paper is to construct a relativistic supersymmetry algebra in the generalized Schrödinger picture which has been proposed in [1]. In this generalization the analogue of Schrödinger operators of the particle are independent of both the time and the space coordinate, t and $\mathbf{x}$. The derivatives ∂_t and $\nabla_{\mathbf{x}}$ cannot be presented in these operators. This picture is based on the principal series of the unitary representations of the Lorentz group. The non-unitary representations are not useful in the generalized Schrödinger picture. For a supersymmetric model in this approach it is necessary to develop a new mathematical formalism in which the supersymmetry generators are expressed in terms of the principal series of the space-time independent representations of the Lorentz group.

The principal series correspond to the eigenvalues $1+\alpha^2-\lambda^2$, $(0\le\alpha<\infty,\quad \lambda=-s,...,s,\quad s=\text{spin})$ of the first $C_1=\mathbf{N}^2-\mathbf{J}^2$ ($\mathbf{N}$, $\mathbf{J}$ are boost and rotation generators) and the eigenvalues $\alpha\lambda$ of the second Casimir operator $C_2=\mathbf{N}\cdot\mathbf{J}$ of the Lorentz algebra [2–4]. For a particle with spin 0 as the eigenfunctions of C_1 in the momentum representation ($\mathbf{p}$ = momentum, $p_0=\sqrt{m^2+\mathbf{p}^2}$, m = mass) one can choose the functions (we use here the notation of [1])

$$\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n})=\frac{1}{(2\pi)^{3/2}}[(pn)/m]^{-1+\mathrm{i}\alpha}, \tag{1.1}$$

where n is a vector on the light-cone $n_0^2-\mathbf{n}^2=0$. For a particle with spin 1/2 the eigensolutions of both operators C_1 and C_2 may be written in the form

$$\widetilde{\xi}^{(1/2)}(\mathbf{p},\alpha,\mathbf{n})=D^{(1/2)}(\mathbf{p},\mathbf{n})\times D^{(1/2)}(\mathbf{n})\,\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n}), \tag{1.2}$$

where

$$D^{(1/2)}(\mathbf{p},\mathbf{n})=\frac{pn+m-\mathrm{i}\sigma\cdot(\mathbf{p}\times\mathbf{n})}{\sqrt{2(p_0+m)(pn)}},$$
$$D^{\dagger(1/2)}(\mathbf{p},\mathbf{n})\,D^{(1/2)}(\mathbf{p},\mathbf{n})=1, \tag{1.3}$$

and the matrix $D^{(1/2)}(\mathbf{n})$ contains the eigenfunctions of the operator $\sigma\cdot\mathbf{n}$ $(D^{\dagger(1/2)}(\mathbf{n})D^{(1/2)}(\mathbf{n})=1)$.

From the point of view of a supersymmetric model the matrices $D^{(1/2)}(\mathbf{p},\mathbf{n})\,D^{(1/2)}(\mathbf{n})$ in (1.2) and $D^{\dagger(1/2)}(\mathbf{n})\times D^{\dagger(1/2)}(\mathbf{p},\mathbf{n})$ in

$$D^{\dagger(1/2)}(\mathbf{n})D^{\dagger(1/2)}(\mathbf{p},\mathbf{n})\widetilde{\xi}^{(1/2)}(\mathbf{p},\alpha,\mathbf{n})=\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n}), \tag{1.4}$$

may be regarded as matrices which realize supersymmetry transformations. In this paper we use these matrices to construct a supersymmetry algebra in terms of the group parameter α and the vector on the light-cone $\mathbf{n}$. This paper is set out as follows. First we quote the necessary results from the Poincaré algebra for the particles with spin 0 and spin 1/2 in the $\alpha\,\mathbf{n}$ representation. In Sect. 3 in this representation the supersymmetry generators are constructed. In Sect. 4 the massless case is considered. In this section the supersymmetry generators will be used for the construction of the Hamilton and momentum operators for the massless particles with spin 0 and spin 1/2 in the $\alpha\,\mathbf{n}$ representation.

2 The Poincaré algebra

The plane waves $\sim\exp[-\mathrm{i}xp]$ in the states in the generalized Schrödinger picture appear in different representations. There is no $\mathbf{x}$ representation. Consequently, the spatial derivative $-\mathrm{i}\nabla_{\mathbf{x}}$ cannot be used to construct the Hamilton and the momentum operators of the particle. For these operators in this approach one must use a space-time independent representation. Here for the massive rel-

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ativistic particles with spin 0 and spin 1/2 we use the following operators. In [5] it was shown that the functions ($\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$)

$$\xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n}) = \frac{1}{(2\pi)^{3/2}}[(pn)/m]^{-1-\mathrm{i}\alpha} \tag{2.1}$$

are the eigenfunctions of the differential–difference operators ($\mathbf{L}(\mathbf{n}) := \mathbf{L}$)

$$H^{(0)} = m\left[\cosh\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right) + \frac{\mathrm{i}}{\alpha}\sinh\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right) + \frac{\mathbf{L}^2}{2\alpha^2}\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right)\right], \tag{2.2}$$

$$\mathbf{P}^{(0)} = \mathbf{n}\left[H^{(0)} - m\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right)\right] - m\frac{\mathbf{n}\times\mathbf{L}}{\alpha}\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right). \tag{2.3}$$

These operators satisfy the conditions

$$H^{(0)}\,\xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n}) = p_0\,\xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n}),$$

$$\mathbf{P}^{(0)}\,\xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n}) = \mathbf{p}\,\xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n}). \tag{2.4}$$

In [6], in order to construct the Hamilton $H^{(1/2)}$ and the momentum operators $\mathbf{P}^{(1/2)}$ for a relativistic particle with spin 1/2 in the $\alpha\,\mathbf{n}$ representation the functions ($\xi^{(1/2)}(\mathbf{p}, \alpha, \mathbf{n})$ are the eigenfunctions of C_1)

$$\xi^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}) = D^{\dagger(1/2)}(\mathbf{p}, \mathbf{n})\xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n}), \tag{2.5}$$

and the conditions

$$H^{(1/2)}\,\xi^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}) = p_0\,\xi^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}),$$
$$\mathbf{P}^{(1/2)}\,\xi^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}) = \mathbf{p}\,\xi^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}), \tag{2.6}$$

were used. The operators $H^{(1/2)}$ and $\mathbf{P}^{(1/2)}$ have the form ($\mathbf{J}^{(1/2)} = \mathbf{L} + \sigma/2$)

$$H^{(1/2)} = \frac{m}{2}\left[\left(\frac{\alpha(\alpha+\imath) + (\mathbf{J}^{(1/2)})^2}{(\alpha^2+\frac{1}{4})}\right)\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right) + \frac{\alpha - \frac{3\mathrm{i}}{2}}{(\alpha - \frac{\mathrm{i}}{2})}\exp\left(-\mathrm{i}\frac{\partial}{\partial\alpha}\right) - \frac{\sigma\cdot\mathbf{L}+1}{(\alpha^2+\frac{1}{4})}\right], \tag{2.7}$$

$$\mathbf{P}^{(1/2)} = \mathbf{n}\left[H^{(1/2)} - m\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right)\right] + m\left[\frac{\mathbf{n}\times\sigma}{2(\alpha+\mathrm{i}/2)} - \frac{2\alpha(\mathbf{n}\times\mathbf{L}) + (\alpha - \mathrm{i}/2)\mathbf{n}\times\sigma + (\mathbf{n}\sigma)\mathbf{L}}{2(\alpha^2+1/4)} \times\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right)\right]. \tag{2.8}$$

If in addition to $H^{(s)}$ and $\mathbf{P}^{(s)}$ ($s = 0, 1/2$) we use the operators of the Lorentz algebra

$$\mathbf{J}^{(0)} := \mathbf{L},$$
$$\mathbf{N}^{(0)} = \alpha\mathbf{n} + (\mathbf{n}\times\mathbf{L} - \mathbf{L}\times\mathbf{n})/2, \tag{2.9}$$
$$\mathbf{J}^{(1/2)} = \mathbf{L} + \frac{\sigma}{2},$$
$$\mathbf{N}^{(1/2)} = \alpha\mathbf{n} + (\mathbf{n}\times\mathbf{J}^{(1/2)} - \mathbf{J}^{(1/2)}\times\mathbf{n})/2, \tag{2.10}$$

then we have the Poincaré algebra in the spacetime independent $\alpha\,\mathbf{n}$ representation

$$[N_i^{(s)}, P_j^{(s)}] = \imath\delta_{ij}H^{(s)}, \quad [P_i^{(s)}, H^{(s)}] = 0,$$
$$[H^{(s)}, N_i^{(s)}] = -\imath P_i^{(s)}, \tag{2.11}$$
$$[P_i^{(s)}, P_j^{(s)}] = 0, \quad [J_i^{(s)}, H^{(s)}] = 0,$$
$$[P_i^{(s)}, J_j^{(s)}] = \imath\epsilon_{ijk}P_k^{(s)}, \tag{2.12}$$
$$[J_i^{(s)}, J_j^{(s)}] = \imath\epsilon_{ijk}J_k^{(s)}, \quad [N_i^{(s)}, N_j^{(s)}] = -\imath\epsilon_{ijk}J_k^{(s)},$$
$$[N_i^{(s)}, J_j^{(s)}] = \imath\epsilon_{ijk}N_k^{(s)}. \tag{2.13}$$

For this reason the operators $H^{(0)}, \mathbf{P}^{(0)}$ and $H^{(1/2)}$, $\mathbf{P}^{(1/2)}$ in the generalized Schrödinger picture can be identified with the Hamilton and momentum operators for the massive relativistic particles with spin 0 and spin 1/2, respectively.

In order to define the Hamilton and the momentum operators for the spin-1/2 particle one can also use the functions

$$\widetilde{\xi}^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}) = D^{\dagger(1/2)}(\mathbf{n})\xi^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}). \tag{2.14}$$

On the basis of these functions

$$\widetilde{H}^{(1/2)}\,\widetilde{\xi}^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}) = p_0\,\widetilde{\xi}^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}),$$
$$\widetilde{\mathbf{P}}^{(1/2)}\,\widetilde{\xi}^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}) = \mathbf{p}\,\widetilde{\xi}^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}), \tag{2.15}$$

where

$$\widetilde{H}^{(1/2)} = D^{\dagger(1/2)}(\mathbf{n})H^{(1/2)}D^{(1/2)}(\mathbf{n}),$$
$$\widetilde{\mathbf{P}}^{(1/2)} = D^{\dagger(1/2)}(\mathbf{n})\mathbf{P}^{(1/2)}D^{(1/2)}(\mathbf{n}). \tag{2.16}$$

In the Poincaré algebra in this case instead of $\mathbf{J}^{(1/2)}, \mathbf{N}^{(1/2)}$ we have

$$\widetilde{\mathbf{J}}^{(1/2)} = D^{\dagger(1/2)}(\mathbf{n})\mathbf{J}^{(1/2)}D^{(1/2)}(\mathbf{n}),$$
$$\widetilde{\mathbf{N}}^{(1/2)} = D^{\dagger(1/2)}(\mathbf{n})\mathbf{N}^{(1/2)}D^{(1/2)}(\mathbf{n}). \tag{2.17}$$

Below we use $H^{(0)}$, $\mathbf{P}^{(0)}$, $\mathbf{J}^{(0)}, \mathbf{N}^{(0)}$ and $\widetilde{H}^{(1/2)}$, $\widetilde{\mathbf{P}}^{(1/2)}$, $\widetilde{\mathbf{J}}^{(1/2)}$, $\widetilde{\mathbf{N}}^{(1/2)}$ to construct the supersymmetry generators.

3 Supersymmetry generators

In [6] the Hamilton operator $H^{(1/2)}$ was constructed with the help of the operator

$$B = \sqrt{m}\left[2\cosh\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right) - \frac{\mathrm{i}\sigma\cdot\mathbf{L}}{(\alpha - \mathrm{i}/2)}\exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right)\right]. \tag{3.1}$$

This operator was obtained by means of replacing in the matrix $(\sqrt{2(p_0+m)})D^{\dagger(1/2)}(\mathbf{p}, \mathbf{n})$ the quantity p_0 by $H^{(0)}$ and the quantities $\mathbf{p}$ by $\mathbf{P}^{(0)}$:

$$\xi^{\dagger(1/2)}(\mathbf{p}, \alpha, \mathbf{n}) = B\frac{\xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n})}{\sqrt{2(p_0+m)}}. \tag{3.2}$$

Another operator for which

$$KB = 2(H^{(0)} + m), \quad BK = 2(H^{(1/2)} + m), \tag{3.3}$$

has the form

$$K = \sqrt{m}\left[2\cosh\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right) + \frac{2\mathrm{i}}{\alpha}\sinh\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right) + \frac{\mathrm{i}\sigma\cdot\mathbf{L}}{\alpha}\exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right)\right]. \tag{3.4}$$

Introducing the operators

$$\widetilde{B} = D^{\dagger(1/2)}(\mathbf{n})B, \quad \widetilde{K} = KD^{(1/2)}(\mathbf{n}), \tag{3.5}$$

we obtain

$$\widetilde{K}\widetilde{B} = 2(H^{(0)} + m), \quad \widetilde{B}\widetilde{K} = 2(\widetilde{H}^{(1/2)} + m). \tag{3.6}$$

Considering the eigenfunctions of $H^{(0)}$ and $\widetilde{H}^{(1/2)}$ as superpartner one can define $\widetilde{B}$ and $\widetilde{K}$ as operators which realize the following supersymmetry transformations:

$$\widetilde{B}\,\xi^{*(0)}(\mathbf{p},\alpha,\mathbf{n}) = \sqrt{2(p_0+m)}\widetilde{\xi}^{\dagger(1/2)}(\mathbf{p},\alpha,\mathbf{n}), \tag{3.7}$$

$$\widetilde{K}\widetilde{\xi}^{\dagger(1/2)}(\mathbf{p},\alpha,\mathbf{n}) = \sqrt{2(p_0+m)}\xi^{*(0)}(\mathbf{p},\alpha,\mathbf{n}). \tag{3.8}$$

Using the anticommuting operators

$$Q_1 = \begin{pmatrix} 0 & \widetilde{K} \\ \widetilde{B} & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & \imath\widetilde{K} \\ -\imath\widetilde{B} & 0 \end{pmatrix},$$

we have the relations

$${Q_1}^2 = {Q_2}^2 = 2(H+m), \tag{3.9}$$

$$[H, Q_1] = 0, \quad [H, Q_2] = 0, \tag{3.10}$$

where

$$H := \begin{pmatrix} H^{(0)} & 0 \\ 0 & \widetilde{H}^{(1/2)} \end{pmatrix}.$$

With the help of Q_1, Q_2 and

$$\mathbf{J} := \begin{pmatrix} \mathbf{J}^{(0)} & 0 \\ 0 & \widetilde{\mathbf{J}}^{(1/2)} \end{pmatrix}, \quad \mathbf{N} := \begin{pmatrix} \mathbf{N}^{(0)} & 0 \\ 0 & \widetilde{\mathbf{N}}^{(1/2)} \end{pmatrix},$$

one can construct other supersymmetry generators. The generators

$$Q_{1i} := [Q_1, J_i], \quad Q_{2i} := [Q_2, J_i], \tag{3.11}$$

may be expressed in the form

$$Q_{1i} = \begin{pmatrix} 0 & \sigma_i\widetilde{K}/2 \\ -\widetilde{B}\sigma_i/2 & 0 \end{pmatrix}, \quad Q_{2i} = \begin{pmatrix} 0 & \imath\sigma_i\widetilde{K}/2 \\ \imath\widetilde{B}\sigma_i/2 & 0 \end{pmatrix},$$

from which we can find that $(r = 1, 2)$

$$\{Q_{ri}, Q_{rk}\} = -(H+m)\delta_{ik}, \tag{3.12}$$

$$[Q_{r1}, J_1] = \frac{1}{4}Q_r, \quad [H, Q_{ri}] = 0, \tag{3.13}$$

and for $i \neq j \neq k$

$$[Q_{ri}, J_j] = \frac{\imath}{2}\epsilon_{ijk}Q_{rk}. \tag{3.14}$$

For the commutators $[Q_1, N_i]$ and $[Q_2, N_i]$ we have

$$[Q_1, N_i] := G_{1i}, \quad [Q_2, N_i] := G_{2i}, \tag{3.15}$$

and we obtain the relations

$$\{G_{ri}, G_{rk}\} = -(H-m)\delta_{ik}, \tag{3.16}$$

$$\{Q_r, G_{ri}\} = -2\imath\begin{pmatrix} P_i^{(0)} & 0 \\ 0 & \widetilde{P}_i^{(1/2)} \end{pmatrix} := -2\mathrm{i}P_i,$$

$$[H, G_{ri}] = 0, \quad [\mathbf{P}, Q_{ri}] = 0, \quad [\mathbf{P}, G_{ri}] = 0,$$
$$[G_{r1}, N_1] = -\frac{1}{4}Q_r, \tag{3.17}$$

and $(i \neq j \neq k)$

$$[G_{ri}, N_j] = -\frac{\imath}{2}\epsilon_{ijk}Q_{rk}, \quad [G_{ri}, J_j] = \frac{\imath}{2}\epsilon_{ijk}G_{rk},$$
$$[Q_{ri}, N_j] = \frac{\imath}{2}\epsilon_{ijk}G_{rk}. \tag{3.18}$$

We write down the explicit form of G_{1i}:

$$G_{1i} = \begin{pmatrix} 0 & \sqrt{m}[(g_{1i})^+_{12} + (g_{1i})^-_{12}]D^{(1/2)}(\mathbf{n}) \\ \sqrt{m}D^{\dagger(1/2)}(\mathbf{n})[(g_{1i})^+_{21} + (g_{1i})^-_{21}] & 0 \end{pmatrix},$$

where

$$(g_{1i})^+_{12} = \frac{\left[\mathrm{i}N_i^{(0)} + (\sigma\times\mathbf{N}^0)_i\right]}{2\alpha}\exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right), \tag{3.19}$$

$$(g_{1i})^+_{21} = \frac{\left[\mathrm{i}N_i^{(1/2)} - n_i/2 - (\sigma\times\mathbf{N}^{(0)})_i\right]}{2(\alpha - \mathrm{i}/2)} \times\exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right), \tag{3.20}$$

and

$$(g_{1i})^-_{12} = \frac{\alpha - \mathrm{i}}{2\alpha}[-\mathrm{i}n_i + (\mathbf{n}\times\sigma)_i]\exp\left(-\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right), \tag{3.21}$$

$$(g_{1i})^-_{21} = \frac{[-\mathrm{i}n_i - (\mathbf{n}\times\sigma)_i]}{2}\exp\left(-\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right). \tag{3.22}$$

The supersymmetry generators which are inroduced in this section give a connection between the states for the massive particles. The mass in the explicit form appear in (3.9), (3.12) and (3.16) in the operator product

$$\exp\left(-\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right)\exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right).$$

For the mass-zero particles we must exclude this term.

4 Massless case

In order to construct the supersymmetry generators for the mass-zero particles we separate $\widetilde{B}$ and $\widetilde{K}$ in two parts corresponding to the operators $\exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right)$ and $\exp\left(-\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right)$, respectively

$$\widetilde{B}^{+} = \sqrt{\mu}D^{\dagger(1/2)}(\mathbf{n})\left[1-\frac{\mathrm{i}\sigma\cdot\mathbf{L}}{(\alpha-\mathrm{i}/2)}\right]\exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right),$$
$$\widetilde{B}^{-} = \sqrt{\mu}D^{\dagger(1/2)}(\mathbf{n})\exp\left(-\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right), \tag{4.1}$$

$$\widetilde{K}^{+} = \sqrt{\mu}\left[\frac{\alpha+\imath+\mathrm{i}\sigma\cdot\mathbf{L}}{\alpha}\right]D^{(1/2)}(\mathbf{n})\exp\left(\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right),$$
$$\widetilde{K}^{-} = \sqrt{\mu}\left[\frac{\alpha-\imath}{\alpha}\right]D^{(1/2)}(\mathbf{n})\exp\left(-\frac{\mathrm{i}}{2}\frac{\partial}{\partial\alpha}\right). \tag{4.2}$$

With the help of these operators one can construct two types of representations of the supersymmetric generators: representations with $\widetilde{B}^{+}$, $\widetilde{K}^{+}$ and representations with $\widetilde{B}^{-}$, $\widetilde{K}^{-}$. In (4.1) and (4.2) we have introduced a constant μ with the dimension of mass in order to deal with dimensional operators.

Let us first start with the case of $\widetilde{B}^{+}$, $\widetilde{K}^{+}$. In Q_r we replace $\widetilde{B}$ by $\widetilde{B}^{+}$ and $\widetilde{K}$ by $\widetilde{K}^{+}$. The results can be obtained from formulas (3.9) to (3.18) by substituting

$$Q_r\to Q_r^{+},\quad Q_{ri}\to Q_{ri}^{+}=[Q_r^{+},J_i],$$
$$G_{ri}\to G_{ri}^{+}=[Q_r^{+},N_i], \tag{4.3}$$

$$Q_{1i}^{+} = \begin{pmatrix} 0 & \sigma_i\widetilde{K}^{+}/2 \\ -\widetilde{B}^{+}\sigma_i/2 & 0 \end{pmatrix},$$
$$G_{1i}^{+} = \begin{pmatrix} 0 & \sqrt{\mu}[(g_{1i})_{12}^{+}]D^{(1/2)}(\mathbf{n}) \\ \sqrt{\mu}D^{\dagger(1/2)}(\mathbf{n})[(g_{1i})_{21}^{+}] & 0 \end{pmatrix}.$$

From

$$Q_r^{+2} = 2\begin{pmatrix} H_0^{+(0)} & 0 \\ 0 & \widetilde{H}_0^{+(1/2)} \end{pmatrix} := 2H_0^{+},$$

and

$$\{Q_r^{+},G_{ri}^{+}\} = -2\imath\begin{pmatrix} P_{0i}^{+(0)} & 0 \\ 0 & \widetilde{P}_{0i}^{+(1/2)} \end{pmatrix} := -2\mathrm{i}P_{0i}^{+},$$

we obtain the operators (the explicit forms of $P_{0i}^{+(0)}$ and $\widetilde{P}_{0i}^{+(1/2)}$ are given in the appendix)

$$H_0^{+(0)} = \mu\left[\frac{\alpha(\alpha+\imath)+\mathbf{L}^2}{2\alpha^2}\right]\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right), \tag{4.4}$$

$$\widetilde{H}_0^{+(1/2)} = \mu\left[\frac{\alpha(\alpha+\imath)+\left(\widetilde{\mathbf{J}^{(\frac{1}{2})}}\right)^2}{2\left(\alpha^2+\frac{1}{4}\right)}\right]\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right), \tag{4.5}$$

which satisfy the conditions

$$(H_0^{+(0)})^2-(\mathbf{P}_0^{+(0)})^2=0,$$
$$(\widetilde{H}_0^{+(1/2)})^2-(\widetilde{\mathbf{P}}_0^{+(1/2)})^2=0, \tag{4.6}$$

and the commutation relations of the Poincaré algebra. Instead of (3.12) and (3.16) we have

$$\{Q_{ri}^{+},Q_{rk}^{+}\}=-H_0^{+}\delta_{ik},\quad \{G_{ri}^{+},G_{rk}^{+}\}=-H_0^{+}\delta_{ik}, \tag{4.7}$$

and one can consider $H_0^{+(s)}$, $\mathbf{P}_0^{+(s)}$ as the Hamilton and momentum operators for the mass-zero particles.

In order to go over to the case with $\widetilde{K}^{-}$ and $\widetilde{B}^{-}$, we must replace in Q_r, Q_{ri}, G_{ri} the generator $\widetilde{B}$ by $\widetilde{B}^{-}$ and the generator $\widetilde{K}$ by $\widetilde{K}^{-}$. From Q_1^{-} and

$$Q_{1i}^{-} = \begin{pmatrix} 0 & \sigma_i\widetilde{K}^{-}/2 \\ -\widetilde{B}^{-}\sigma_i/2 & 0 \end{pmatrix},$$
$$G_{1i}^{-} = \begin{pmatrix} 0 & \sqrt{\mu}[(g_{1i})_{12}^{-}]D^{(1/2)}(\mathbf{n}) \\ \sqrt{\mu}D^{\dagger(1/2)}(\mathbf{n})[(g_{1i})_{21}^{-}] & 0 \end{pmatrix},$$

we obtain

$$Q_1^{-2}=2H_0^{-},\quad \{Q_1^{-},G_{1i}^{-}\}=-2\mathrm{i}P_{0i}^{-}, \tag{4.8}$$

with

$$H_0^{-(0)} = \mu\left[\frac{(\alpha-\imath)}{2\alpha}\right]\exp\left(-\mathrm{i}\frac{\partial}{\partial\alpha}\right),$$
$$\widetilde{H}_0^{-(1/2)} = \mu\left[\frac{\left(\alpha-\frac{3\mathrm{i}}{2}\right)}{2\left(\alpha-\frac{\mathrm{i}}{2}\right)}\right]\exp\left(-\mathrm{i}\frac{\partial}{\partial\alpha}\right), \tag{4.9}$$

$$\mathbf{P}_0^{-(0)}=\mathbf{n}H_0^{-(0)},\quad \widetilde{\mathbf{P}}_0^{-(1/2)}=\mathbf{n}\widetilde{H}_0^{-(1/2)}. \tag{4.10}$$

For these operators we also have the conditions

$$(H_0^{-(0)})^2-(\mathbf{P}_0^{-(0)})^2=0,$$
$$(\widetilde{H}_0^{-(1/2)})^2-(\widetilde{\mathbf{P}}_0^{-(1/2)})^2=0, \tag{4.11}$$

and the commutation relations of the Poincaré algebra. Additionally,

$$\{Q_{ri}^{-},Q_{rk}^{-}\}=-H_0^{-}\delta_{ik},\quad \{G_{ri}^{-},G_{rk}^{-}\}=-H_0^{-}\delta_{ik}, \tag{4.12}$$

$$[H_0^{-},G_{ri}^{-}]=0,\quad [\mathbf{P}_0^{-},Q_{ri}^{-}]=0,\quad [\mathbf{P}_0^{-},G_{ri}^{-}]=0. \tag{4.13}$$

For the eigenfunctions of $H_0^{-(0)}$ and $\mathbf{P}_0^{-(0)}$ we may choose $(-\infty<\gamma<\infty)$

$$\Psi^{-(0)}(\alpha,\mathbf{n},\gamma,\mathbf{n}')=\frac{1}{\sqrt{\pi\alpha}}\exp(-\gamma+\mathrm{i}\alpha\gamma)\delta(\mathbf{n}-\mathbf{n}'). \tag{4.14}$$

Here the eigenvalues of $H_0^{-(0)}$ are determined by $k_0=\mu\frac{e^{\gamma}}{2}$, and the eigenvalues of $\mathbf{P}_0^{-(0)}$ by $\mathbf{k}=k_0\mathbf{n}'$. For the eigenfunctions of $\widetilde{H}_0^{-(1/2)}$ and $\widetilde{\mathbf{P}}_0^{-(1/2)}$ we have

$$\Psi^{-(1/2)}(\alpha,\mathbf{n},\gamma,\mathbf{n}')=\widetilde{B}^{-}\frac{\Psi^{-(0)}(\alpha,\mathbf{n},\gamma,\mathbf{n}')}{\sqrt{2k_0}}. \tag{4.15}$$

With the help of $Q^-_{r,} Q^-_{ri}$ and G^-_{ri}, one can find other eigenfunctions of H^-_0, $\mathbf{P}^-_0$.

If we return to the massive particles with $\widetilde{B}^+ + \widetilde{B}^-$ and $\widetilde{K}^+ + \widetilde{K}^-$, we can find that in this case the mass in (3.9), (3.12) and (3.16) may be expressed through the constant μ for the massless particles. Particularly, for the particles with spin 0 we obtain

$$H^{(0)} = H_0^{+(0)} + H_0^{-(0)}, \tag{4.16}$$

$$m = (\widetilde{K}^+\widetilde{B}^- + \widetilde{K}^-\widetilde{B}^+)/2 = \mu. \tag{4.17}$$

5 Conclusion

We have shown that in the generalized Schrödinger picture a relativistic superalgebra may be constructed by using the principal series of the unitary representation of the Lorentz group. In the construction the Poincaré algebra for the massive particles with spin 0 and spin 1/2 in terms of the invariant parameter α and the vector on the light-cone $\mathbf{n}$ was used. In this representation we found the explicit form of the supersymmetry generators. For the massless case we have used two types of representations of the supersymmetry generators to construct new Hamilton and momentum operators for such particles with spin 0 and spin 1/2.

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Appendix

Momentum operators

The momentum operators for the massless case in (4.6) may be written as follows:

$$\mathbf{P}_0^{+(0)} = \mathbf{n}H_0^{+(0)} - \mu\frac{1}{\alpha}\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right)\mathbf{N}^{(0)}, \tag{A1}$$

$$\widetilde{\mathbf{P}}_0^{+(1/2)} = \mathbf{n}\left[\widetilde{H}_0^{+(1/2)} - \mu\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right)\right] - \mu D^{\dagger(1/2)}(\mathbf{n}) \times\frac{2\alpha(\mathbf{n}\times\mathbf{L}) + (\alpha - \mathrm{i}/2)\mathbf{n}\times\sigma + (\mathbf{n}\sigma)\mathbf{L}}{2(\alpha^2+1/4)} \times D^{(1/2)}(\mathbf{n})\exp\left(\mathrm{i}\frac{\partial}{\partial\alpha}\right)]. \tag{A2}$$

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